Non–analytic strong–to–weak coupling transitions in $SU(\infty)$ gauge theory

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Contents

Introduction

There are several transitions in $SU(N)$ gauge theories which are in some sense strong to weak coupling transitions. These are:

- The Gross–Witten transition at $N = \infty$ in $D = 1 + 1$.
- A non–analyticity in Wilson loop eigenvalue spectra at $N = \infty$ in the $D = 1 + 1$ continuum theory.
- The bulk transition for $N \geq 5$ in $D = 3 + 1$.

At all of these a gap opens in the plaquette or Wilson loop eigenvalue spectrum.

Is there also a continuum transition in $D = 3 + 1$? Does it have a connection to the rapid crossover form perturbative to non–perturbative physics in QCD?

Gross–Witten transition

$D = 1 + 1$ $SU(N)$ lattice gauge theory can be solved exactly. It reduces to an integral over a single plaquette. For $N = \infty$ this is evaluated by considering the eigenvalue density $\rho(\alpha)$.

- For strong coupling the trace of the plaquette is small, so the eigenvalue density is nearly uniform.
- For weak coupling the trace $\rightarrow 1$, so the distribution becomes peaked around $\alpha = 0$.

In between there is a third–order phase transition at $\gamma \equiv \frac{\beta}{2N^2} = 0.5$. This is a strong to weak coupling transition. At the transition a gap opens in the density of eigenvalues — on the weak coupling side the density is zero for $|\alpha| > \alpha_c$.

[*D. Gross, E. Witten, Phys. Rev. D21 (1980) 446*]

Wilson loops

The Gross–Witten transition happens at a critical value of the bare coupling. As we change the bare coupling, we change the lattice spacing — so the transition happens when the plaquette passes through a critical length scale.

What happens to other Wilson loops when they pass through this length scale?

In fact there is a similar transition in the continuum theory. A gap opens when the trace is e^{-2}. This happens at a critical area

$$A_c = \frac{8}{g^2 N}. \tag{1}$$

The physical significance of this transition is unclear.

[*B. Durhuus and P. Olesen, Nucl. Phys. B184 (1981) 461*]

[*A. Bassetto, L. Griguolo and F. Vian, Nucl. Phys. B559 (1999) 563*]

3+1 dimensions

In $D = 3 + 1$, there is a strong first–order phase transition for $N \geq 5$. This is also a strong to weak coupling transition. For $N = \infty$ a gap opens in the plaquette eigenvalue spectrum.

Is there a transition for Wilson loops, like the one in $D = 1 + 1$? If so this could explain the observed rapid transition from perturbative to non–perturbative physics.

[*R. Narayanan and H. Neuberger, hep-lat/0501031, hep-lat/0509014*]

It has also been suggested that by matching the spectra on either side of the non–analyticity via RMT one could obtain non–perturbative parameters in terms of perturbative ones.

[*R. Narayanan and H. Neuberger, hep-th/0601210*]

Matching eigenvalue spectra

There is a remarkable matching between eigenvalue spectra in 1+1, 2+1 and 3+1 dimensions:

- Take $n \times n$ loops in D and D' dimensions in $SU(N)$.
- Adjust the couplings so the traces match.

Then the eigenvalue densities will match.

- Also appears to work for smeared Wilson loops.

If exact, implies non–analyticity in $D = 1 + 1$ must also be present in $D = 2 + 1$ and $D = 3 + 1$.

- Is the matching exact?
- Does the non–analyticity occur at a fixed length scale in the continuum limit?

Methods

We know that any violations of the matching must be small. So need high statistics and control over systematic errors.

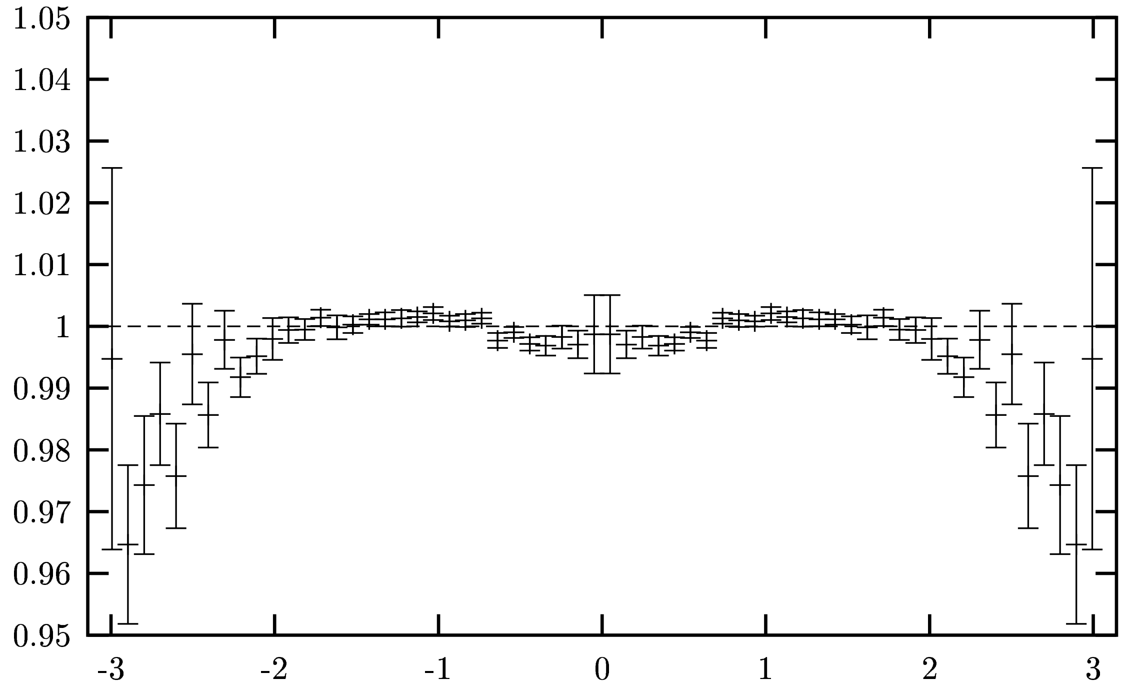
- Match traces exactly by reweighting.
- Generate histograms of eigenvalue densities.
- Calculate Fourier components and moments.
- Compare all these.

Also check for finite–volume effects.

Results

Plaquettes do not match exactly.

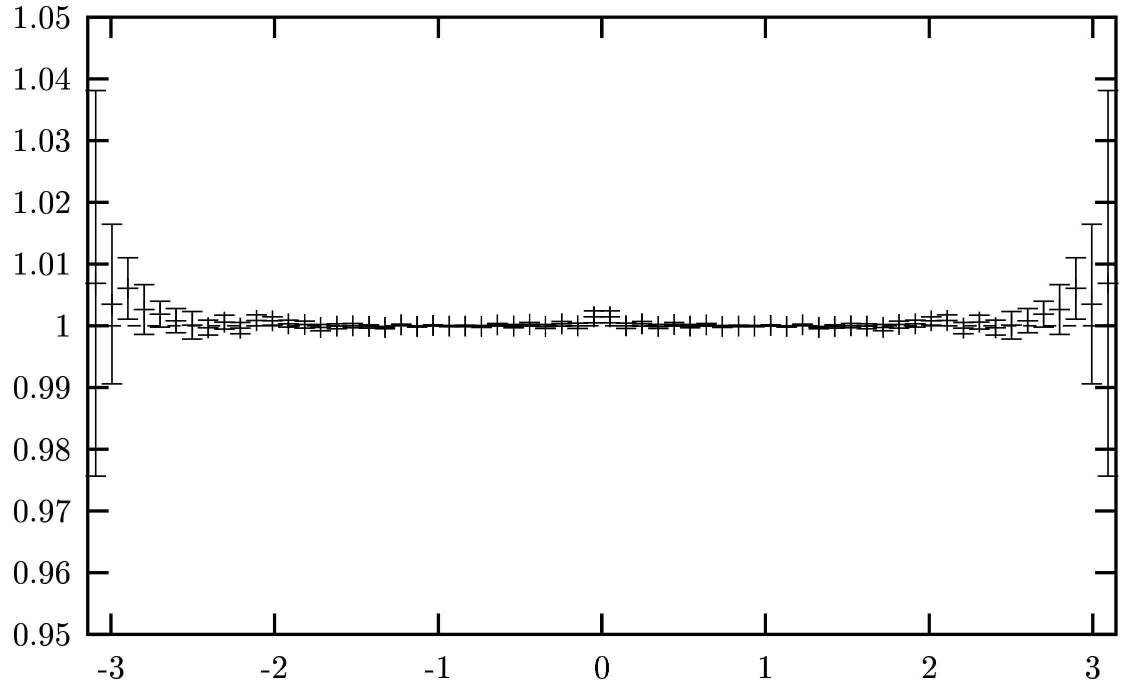
- For example, plaquettes in 1+1d and 2+1d with trace 0.5:



Ratio of plaquette eigenvalue densities in 2+1 and 1+1 dimensions in SU(2) with trace 0.5

Results

But for larger loops the differences decrease:



Ratio of 4×4 loop eigenvalue densities in 2+1 and 1+1 dimensions in SU(2) with trace 0.5

Looks like matching may be exact in continuum limit.

Can we understand this analytically?

Strong coupling

Fourier components of eigenvalue densities are just traces of powers of the Wilson loop:

$$F_k = \frac{1}{N}\mathrm{Re}\ \mathrm{Tr}(W^k). \tag{2}$$

Consider strong–coupling expansion of $\mathrm{Tr}(W^k)$:

- Leading term comes from tiling the loop with k sheets of plaquettes.
- Same in all dimensions.

So if traces matched, densities should match.

Higher orders in strong–coupling:

- 'Bumps' in sheets of plaquettes will become important.
- Expect violations of matching.

Weak coupling

Traces of powers of a Wilson loop can be expressed in terms of traces in different representations.

To two loops, Wilson loops satisfy Casimir Scaling:

$$\frac{1}{N}\mathrm{Tr}_R W = e^{-C_R f_D(g^2)}. \tag{3}$$

- Matching traces means f_D match.
- Then traces in all representations will match, so densities will match.

Continuum limit

In 2+1 and 3+1 dimensions, there is a 'perimeter term' contribution to Wilson loops:

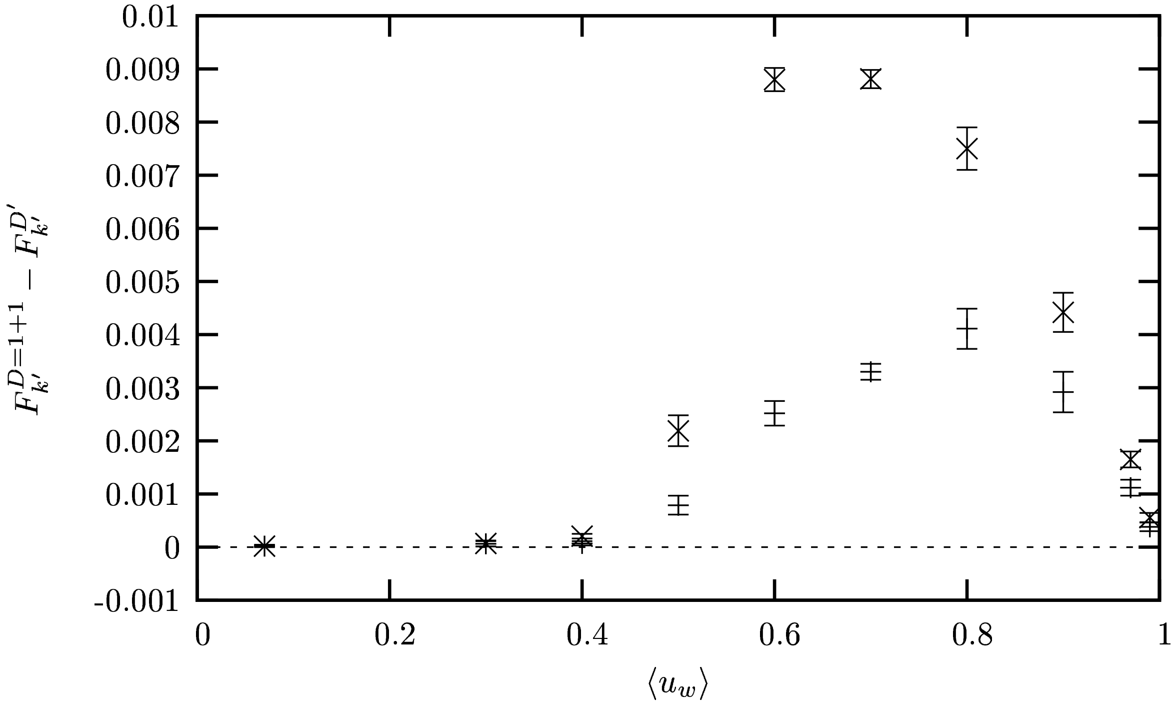
$$\langle u_w \rangle \propto \begin{cases} \exp(caLg^2N\ln(ag^2N) - \sigma a^2L^2) & D = 2+1 \\ \exp(cg^2NL - \sigma a^2L^2) & D = 3+1 \end{cases} \tag{4}$$

So boundary of weak–coupling regime will shift to lower traces in continuum limit.

- Casimir Scaling on weak–coupling side of this boundary.
- Casimir Scaling at all couplings in 1+1 dimensions.

So expect matching on weak–coupling side of boundary. For large enough L will see matching at *all* values of the trace.

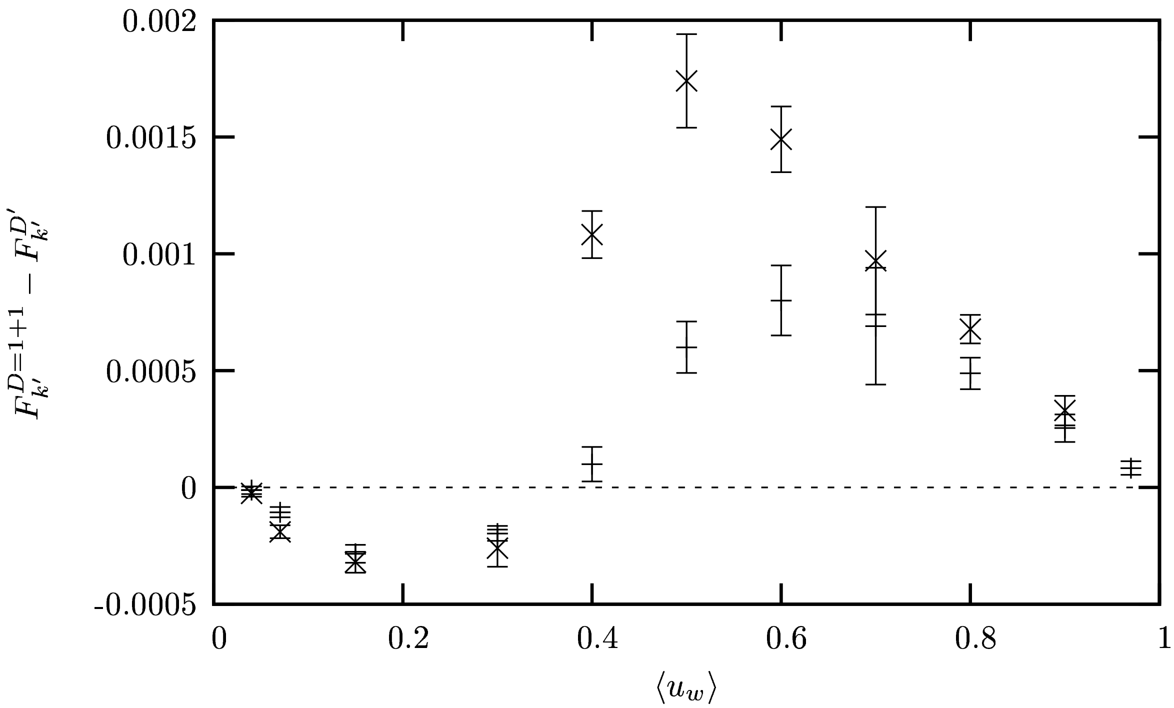
Plaquette in $SU(2)$



Differences between $F_{k'}$ for the plaquette in 1+1 dimensions and D' dimensions.

- Differences go to zero as $\beta \to 0$ and $\beta \to \infty$, as expected.
- Differences at intermediate couplings non–zero, but small.
- Differences very small throughout strong coupling.

2×2 **Wilson loops in** $SU(2)$



Differences between $F_{k'}$ for 2×2 Wilson loops in 1+1 dimensions and D' dimensions.

- Differences shifted to smaller trace.
- Differences much smaller.

4×4 loops: differences only just detectable.

Non–analyticity at $N = \infty$

Pattern of violations is consistent with strong–coupling and weak–coupling analysis.

In continuum limit, will get matching at all values of the trace.

In particular, the non–analyticity in 1+1 dimensions will also be present in $D = 2 + 1$ and $D = 3 + 1$.

But due to perimeter term, it will occur at UV length scales.

Non–analyticity at $N = \infty$

What about smearing?

- Smearing would reduce perimeter term, so non–analyticity could occur at a physical scale.
- But scale would depend on details of smearing.
- Wilson loops would have an area term, which does not obey Casimir Scaling.
- So matching would be violated, and don't expect (same) non–analyticity.

Can't use this to match perturbative and non–perturbative physics?

Conclusions

- Eigenvalue densities match between different dimensions in the strong–coupling and weak–coupling limits.
- Violations are small at intermediate couplings.
- There is matching at all values of the trace in the continuum limit.
- There is a non–analyticity at $N = \infty$ in $D = 3 + 1$, but at a UV length scale.