Point Form Quantum Field Theory and Applications
ECT* Workshop

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1 Overview of Point Form Quantum Theory

-Definition of a relativistic quantum theory: A set of operators acting on a generalized Hilbert (Fock) space which satisfy the commutation relations of the Poincare algebra. Valid for both finite and infinite degree of freedom systems. Several ways of doing this which go under the names of instant, front and point forms of dynamics.

-Point form relativistic quantum theory: Dynamics resides in the four-momentum operator P^μ and the Lorentz generators are kinematic. The Lorentz part of the Poincare commutation relations can be integrated to give

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ U_\Lambda P^\mu U_\Lambda^{-1} &= (\Lambda^{-1})^\mu_\nu P^\nu, \end{aligned}$$

where U_Λ is the unitary operator representing the Lorentz transformation Λ on the Hilbert space.

-Given four-momentum operator satisfying point form equations, want to solve

$$P^\mu| >= p^\mu| >,$$

to get vacuum, bound and scattering solutions.

-For noninteracting systems, start with massive irreps of Poincare group, introduce the four-velocity, $v := \frac{p}{m}$, the four-momentum divided by mass, as the natural variable for point form dynamics.

Under a Poincare transformation,

$$\begin{aligned} U_a|v,\sigma> &= e^{ip\cdot a}|v,\sigma> \\ U_\Lambda|v,\sigma> &= \sum|\Lambda v,\sigma^{'}> D^j_{\sigma^{'},\sigma}(R_W(v,\Lambda)), \end{aligned}$$

where a is a four-translation, $D^j_{\sigma^{'},\sigma}()$ and is an $SU(2)$ matrix element for spin j.

$R_W(v,\Lambda)$ is a Wigner rotation, an element of the rotation group $SO(3)$ defined by

$$R_W(v,\Lambda):\ =\ B^{-1}(\Lambda v)\Lambda B(v).$$

$B(v)$ is a boost, a Lorentz transformation satisfying $v = B(v)v^{rest}$, with $v^{rest} = (1,0,0,0)$. Here boosts are canonical spin boosts.

-Introduce classical irreducible fields over Minkowski space-time, with differentiation inner product (with W. Schweiger and collaborators, Ann. Phys. 323 (2008)1362).

-Example: scalar particle, with wave-function $\phi(v) \in L^2(R^3)$.

$$\begin{aligned}
||\phi||^2 &= \int dv|\phi(v)|^2; dv := \frac{d^3v}{(2\pi)^{3/2}2v_0} \\
\psi(x) &= \int dve^{-imv\cdot x}\phi(v) \\
||\psi||^2 &= \int dx^\mu(\frac{\partial\psi^*}{\partial x^\mu}\psi - \psi^*\frac{\partial\psi}{\partial x^\mu}) \\
&= ||\phi||^2.
\end{aligned}$$

Note that the integration on space-time $dx^\mu := d^4x\delta(x\cdot x-1)\theta(x_0)x^\mu$ is over the forward hyperboloid, typical of the point form.

-Many-particle theory: Introduce creation and annihilation operators with the usual commutation relations for fermions ($a(v,\sigma,\tau)$, antifermions ($b(v,\sigma,\tau)$ and bosons ($c(v,\alpha,i)$. Products of bilinears $a^\dagger a, a^\dagger b^\dagger, ba, bb^\dagger$ along with products of boson creation and annihilation operators form the algebra of operators $\mathcal{A}$. Acting on the Fock vacuum generates the usual Fock space.

The free four-momentum operator is made from this algebra of operators:

$$\begin{aligned} P^\mu_{free} :&= m\sum\int dv v^\mu(a^\dagger(v,\sigma,\tau)a(v,\sigma,\tau) \\ &-b(v,\sigma,\tau)b^\dagger(v,\sigma,\tau)) + \kappa(c^\dagger(v,\alpha,i)c(v,\alpha,i)), \end{aligned}$$

where κ is a dimensionless relative bare boson mass parameter and m is a constant with the dimensions of mass. τ and i are internal symmetry variables. It is easily checked that the free four-momentum operator satisfies the point form equations.

-Photons, Gluons and Gravitons in point form: massless particles transform as representations of the little group E(2); for photons and gluons choose (contra Weinberg) the four dimensional nonunitary irrep of E(2) to get four polarization degrees of freedom (labeled ρ). The standard four-vector $k^{st} = (1, 0, 0, 1)$ leaves E(2) invariant; a helicity boost, $B(k) := R(\hat{k})\Lambda_z(|\vec{k}|)$, to a four-momentum k then generates a gluon state with transformation properties

$$
\begin{aligned}
U_{e_2}|k^{st}\rho, a> &= \sum |k^{st}\rho^{'}, a > \Lambda_{\rho^{'}\rho}(e_2),\\
|k, \rho, a >: &= U_{B(k)}|k^{st}, \rho, a >,\\
U_{\Lambda}|k, \rho, a > &= U_{\Lambda}U_{B(k)}|k^{st}, \rho, a >\\
&= \sum |\Lambda k, \rho^{'}, a > \Lambda_{\rho^{'}\rho}(e_W),\\
U_c|k, \rho, a > &= \sum |k, \rho, a^{'} > D_{a^{'}a}(c),
\end{aligned}
$$

where $\Lambda(e_W) = B^{-1}(\Lambda k)\Lambda B(k)$ is a euclidean Wigner "rotation", c is an element of the internal symmetry (color) group and $a, a^{'}$ are color indices.

-A Lorentz invariant gluon inner product is given by

$$||\phi||^2 = -\sum \int dk \eta_{\rho\rho} |\phi(k, \rho, a)|^2.$$

-Origin of gauge degrees of freedom: inner product is not positive definite, so impose Lorentz invariant auxiliary condition $\phi(k, 0, a) = \phi(k, 3, a)$. Then only two polarizations are physical, as required for massless particles.

-Classical gluon field is defined by

$$G_a^\mu(x) = \int dk B^{\mu\rho}(k) \phi(k, \rho, a) e^{-ik \cdot x},$$

with a norm given by differentiation inner product over forward hyperboloid. Note that polarization matrix is a Lorentz boost.

Then can show $||G||^2 = ||\phi||^2$.

-Many gluon states: Introduce gluon creation and annihilation operators such that

$$
\begin{aligned}
|k,\rho,a> &= g^{\dagger}(k,\rho,a)|0> \\
g(k,\rho,a)|0> &= 0, \forall k,\rho,a \\
[g(k,\rho,a), g^{\dagger}(k^{'},\rho^{'},a^{'})] &= -\eta_{\rho,\rho^{'}} k_0 \delta^3(\vec{k}-\vec{k}^{'})\delta_{aa^{'}} \\
U_{\Lambda} g(k,\rho,a) U_{\Lambda}^{-1} &= \sum g(\Lambda k,\rho^{'},a)\Lambda_{\rho,\rho^{'}}(e_W) \\
U_c g(k,\rho,a) U_c^{-1} &= g(k,\rho,a^{'}) D_{a^{'}a}(c) \\
P^{\mu}_{free} &= -\sum \int \frac{d^3k}{k_0} k^{\mu} g^{\dagger}(k,\rho,a)\eta^{\rho\rho} g(k,\rho,a).
\end{aligned}
$$

-The auxiliary condition eliminating the 0 and 3 components on the physical many-body Fock space is the annihilation operator condition,

$$\sum k^{st}_{\rho} \eta^{\rho\rho} g(k,\rho,a)|\phi >= 0,$$

which is Lorentz invariant.

-Gauge transformations change 0 and 3 components only, given by

$$
\begin{aligned}
g(k,\rho,a) \rightarrow g^{'}(k,\rho,a) &= g(k,\rho,a) + k^{st}_{\rho} f(k,a) \\
g^{\dagger}(k,\rho,a) \rightarrow g^{'\dagger}(k,\rho,a) &= g^{\dagger}(k,\rho,a),
\end{aligned}
$$

and leaves commutation relations and auxiliary condition unchanged.

-The free gluon field is

$$\begin{aligned} G^\mu_a(x) &= \int dk B^{\mu\rho}(k)(e^{-ik\cdot x} g(k,\rho,a) + e^{ik\cdot x} g^\dagger(k,\rho,a)) \\ \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} G^\mu_a &= 0. \end{aligned}$$

-Under a gauge transformation get

$$\begin{aligned} G^\mu_a(x) \rightarrow G'^{\mu}_a(x) &= G^\mu_a(x) + \frac{\partial \tilde{f}(x,a)}{\partial x_\mu} I; \\ \tilde{f}(x,a) &= \int \frac{d^3k}{2k_0} e^{-ik\cdot x} f(k,a), \end{aligned}$$

which preserves the Lorentz gauge on the physical Fock space:

$$\begin{aligned} \partial G^+_{\mu,a}(x)/\partial x_\mu |\phi> &= i \int dk k^\mu B_{\mu\rho}(k) \eta_{\rho,\rho} e^{-ik\cdot x} g(k,\rho,a)|\phi> \\ &= i \sum \int dk e^{-ik\cdot x} k^{st}_\rho \eta_{\rho\rho} g(k,\rho,a)|\phi> \\ &= 0. \end{aligned}$$

2 Point Form Quantum Field Theory

-Massive free Quantum Fields (with W. Schweiger and collaborators): Use mapping between irreps of Poincare group and classical irreducible fields to define (for example spin 1/2 and 0)

$$\begin{aligned}\Psi_\tau(x) &= \sum\int dv(a(v,\sigma,\tau)u_\sigma(v)e^{-imv\cdot x} + b^\dagger(v,\sigma,\tau)v_\sigma(v)e^{imv\cdot x}) \\ \phi_i(x) &= \int dv(c(v,i)e^{-imv\cdot x} + c^\dagger(v,i)e^{imv\cdot x}).\end{aligned}$$

-Define usual free Lagrangian and get P^μ_{free} from stress energy tensor,

$$\begin{aligned}P^\mu_{free} &= \int d^4x\delta(x\cdot x-1)\theta(x_0)x_\nu T^{\mu\nu}_{free} \\ &= \int dx_\nu T^{\mu\nu}_{free}.\end{aligned}$$

-Example: Free charged scalar field,

$$\begin{aligned}\mathcal{L} &= \partial^\mu\phi^\dagger(x)\partial_\mu\phi(x) - m^2\phi^\dagger(x)\phi(x); \\ T^{\mu\nu} &= \partial^\mu\phi^\dagger(x)\partial^\nu\phi(x) + \partial^\nu\phi^\dagger(x)\partial^\mu\phi(x) - \eta^{\mu\nu}\mathcal{L}.\end{aligned}$$

-Claim: The free four-momentum operator obtained from Lagrangian is the same as the one obtained from irreps of Poincare group.

-Can extend to massless particles like gluons, free Lagrangian (for example $\mathcal{L} = F^{\alpha\beta}F_{\alpha\beta}$) generates stress energy tensor that gives free four-momentum operator.

-Claim: It agrees with free four-momentum operator from Poincare representations on physical Fock space.

-Application: Self coupling of gluon field:

$$\begin{aligned}
F_a^{\mu\nu}(x) &= \frac{\partial G_a^\nu}{\partial x_\mu} - \frac{\partial G_a^\mu}{\partial x_\nu} - \alpha c_{a,b,c} G_b^\mu(x) G_c^\nu(x), \\
T^{\mu\nu}(x) &= F_a^{\alpha\beta}(x)\eta_{\beta\beta'}[\eta^\mu_{\alpha'}\eta^\nu_\alpha + \eta^\nu_{\alpha'}\eta^\mu_\alpha - \frac{1}{2}\eta^{\mu\nu}\eta_{\alpha\alpha'}]F^{\alpha'\beta'}, \\
P^\mu_{gluon} &= \int_{hyper} dx_\nu T^{\mu\nu},
\end{aligned}$$

where $c_{a,b,c}$ are the color structure constants and α is the strong bare coupling constant.

-Vacuum Problem: Find a state $|\Omega>$ such that it carries a one dimensional representation of the Poincare group and is invariant under internal symmetries:

$$\begin{aligned} P^\mu|\Omega> &= 0 \\ U_\Lambda|\Omega> &= |\Omega> \\ U_c|\Omega> &= |\Omega>, \end{aligned}$$

where c is an element of the internal symmetry group.

-Three points to make about vacuum:

1. Cannot add a multiple of the identity operator to P^μ without violating point form equations.
2. It suffices to calculate $P^0|\Omega>= 0$, for invariance under Lorentz transformations implies $\vec{P}|\Omega>= 0$.
3. Will write $|\Omega>= e^S|0>$, which generates a set of vacuum equations. But $|\Omega>$ is not in Fock space generated by the Fock vacuum $|0>$;

-Problem: How to find gluon vacuum state? Consider simple model with all spatial dimensions suppressed; then have

$$\begin{aligned} g^\dagger(k,\rho,a) &\rightarrow g^\dagger \\ G^\mu_a(x) &\rightarrow g+g^\dagger \qquad (x) \\ \frac{\partial G^\mu_a}{\partial x^\nu} &\rightarrow i(g-g^\dagger) \qquad (p) \\ [g,g^\dagger] &= 1 \\ H &= [i(g-g^\dagger)+\alpha(g+g^\dagger)^2]^2, \\ He^S|0> &= \lambda e^S|0>, \\ e^{-S}[i(g-g^\dagger)+\alpha(g+g^\dagger)^2]e^S & \\ \times e^{-S}[i(g-g^\dagger)+\alpha(g+g^\dagger)^2]e^S|0> &= \lambda|0> \end{aligned}$$

with $\lambda = 0$ for the vacuum. Choose S to be a polynomial in $g+g^\dagger$, then $[g-g^\dagger, S] = 2\dot{S}$ and $e^{-S}[i(g-g^\dagger)+\alpha(g+g^\dagger)^2]e^S = 2i\dot{S}+i(g-g^\dagger+\alpha(g+g^\dagger)^2$; pick $S = \frac{i\alpha}{6}(g+g^\dagger)^3$, then interacting terms are cancelled, leaving only free kinetic energy term.

Question: Will this also work for full gluon problem?

-Fermions and Vertex Interactions: A vertex $V(x)$ is a local space-time scalar density operator,

$$\begin{aligned} e^{-iP_{free}\cdot a}V(x)e^{iP_{free}\cdot a} &= V(x+a) \\ U_\Lambda V(x)U_\Lambda^{-1} &= V(\Lambda x) \\ [V(x),V(y)] &= 0,(x-y)^2 spacelike. \end{aligned}$$

-Then the interacting four-momentum operator is defined by

$$\begin{aligned} P_I^\mu &= \int dx^\mu V(x) \\ &= \int d^4x\delta(x\cdot x-1)\theta(x_0)x^\mu V(x) \end{aligned}$$

and satisfies the point form equations. Further, the total four-momentum operator, $P^\mu := P^\mu_{free} + P^\mu_I$ also satisfies the point form equations.

-Example: the pion-nucleon vertex,

$$\begin{aligned} V(x) &= \quad \bar{\Psi}(x) \quad \gamma_5\vec{\tau} \quad \Psi(x) \quad \vec{\phi} \\ &= (a^\dagger + b) \qquad (a+b^\dagger) \quad (c+c^\dagger) \\ &= (a^\dagger a + a^\dagger b^\dagger + ba + bb^\dagger)(c+c^\dagger) \end{aligned}$$

-Write the four-momentum operator for the pi-N vertex model as $P^\mu = P_N^\mu + \tilde{\kappa} P_\pi^\mu + \alpha P_I^\mu$, where the individual four-momentum operators are given by

$$\begin{aligned}
P_N^\mu &= \sum \int \frac{d^3 v}{v^0} v^\mu (a_{v\sigma}^\dagger a_{v\sigma} - b_{v\sigma} b_{v\sigma}^\dagger) \\
&= \mathcal{A}(E^\mu); \\
P_\pi^\mu &= \int \frac{d^3 v}{v^0} v^\mu c_v^\dagger c_v; \\
P_I^\mu &= \int d^4 x \delta(x \cdot x - \tau^2) x^\mu \bar{\Psi}(x) \gamma_5 \Psi(x) \phi(x) \\
&= \int \frac{d^3 v}{v^0} (\mathcal{A}(X_v^\mu) c_v + \mathcal{A}(X_v^{\mu\dagger}) c_v^\dagger).
\end{aligned}$$

That the integration over x can be rewritten as integrals over a velocity variable can be seen as follows: Define

$$\mathcal{A}(Y) : = \int \frac{d^3 v_1}{v_1^0} \frac{d^3 v_2}{v_2^0} (a_{v_1\sigma_1}^\dagger, b_{v_1\sigma_1}) \left[\begin{array}{cc} Y_{v_1\sigma_1, v_2\sigma_2}^{11} & Y_{v_1\sigma_1, v_2\sigma_2}^{12} \\ Y_{v_1\sigma_1, v_2\sigma_2}^{21} & Y_{v_1\sigma_1, v_2\sigma_2}^{22} \end{array} \right] (a_{v_2\sigma_2}, b_{v_2\sigma_2}^\dagger)^T;$$

Then

$$\begin{aligned}
E^\mu &= \left[\begin{array}{cc} v_1^\mu v_2^0 \delta^3(v_1 - v_2) \delta_{\sigma_1\sigma_2} & 0 \\ 0 & -v_1^\mu v_2^0 \delta^3(v_1 - v_2) \delta_{\sigma_1\sigma_2} \end{array} \right] \\
X_v^\mu &= \int d^4 x \delta(x \cdot x - \tau^2) \theta(x^0) x^\mu \\
& \left[\begin{array}{cc} e^{i(v_1 - v_2 - v) \cdot x} \bar{u}(v_1\sigma_1) \gamma_5 u(v_2\sigma_2) & e^{i(v_1 + v_2 - v) \cdot x} \bar{u}(v_1\sigma_1) \gamma_5 v(v_2\sigma_2 \\ e^{i(-v_1 - v_2 - v) \cdot x} \bar{v}(v_1\sigma_1) \gamma_5 u(v_2\sigma_2) & e^{i(-v_1 + v_2 - v) \cdot x} \bar{v}(v_1\sigma_1) \gamma_5 v(v_2\sigma_2) \end{array} \right]
\end{aligned}$$

-Application: Strong coupling model; divide four-momentum by bare strong coupling constant, neglect free fermion four-momentum, and look at $\mu = 0$ component:

$$P^0 \;=\; \int \frac{d^3v}{v^0}(\kappa v^0 c_v^\dagger c_v + \mathcal{A}(X_v^0)c_v + \mathcal{A}(X_v^{0\dagger})c_v^\dagger.$$

Define automorphism on pion operators:

$$\begin{aligned} C_v :\; &=\; c_v + \kappa\mathcal{A}(X^{0\dagger}), \\ C_v^\dagger :\; &=\; c_v^\dagger + \kappa\mathcal{A}(X_v^0), \\ P^0 \;&=\; \int \frac{d^3v}{v^0}(\kappa v^0 C_v^\dagger C_v - \frac{1}{\kappa v^0}\mathcal{A}(X_v^0)\mathcal{A}(X_v^{0\dagger})) \end{aligned}$$

can diagonalize with Bogoliubov transformation, add fermion kinetic energy perturbatively.

-Application: "Bakamjian-Thomas" like four-momentum(again neglecting fermion four-momentum):

$$P^\mu \;=\; \int \frac{d^3v}{v^0} v^\mu(\kappa c_v^\dagger c_v + \alpha\mathcal{A}(Y_v)c_v + \mathcal{A}(Y_v^\dagger)c_v^\dagger),$$

satisfies point form equations for any Y_v satisfying $[Y_v, Y_{v'}] = [Y_v, Y_{v'}^\dagger] = 0$.

-Point form interaction picture and covariant perturbation theory(with Schweiger and collaborators): Starting with the relativistic Schrodinger equation, can write:

$$\begin{aligned}
i\partial x_\mu \Psi(x) &= (P^\mu_{free} + P^\mu_I)\Psi(x); \\
P^\mu_I(x) : &= e^{iP_{free}\cdot x} P^\mu_I e^{-iP_{free}\cdot x}; \\
\Psi(x) &= U(x, x_0)\Psi(x_0), \\
i\frac{\partial U(x, x_0)}{\partial x_\mu} &= P^\mu_I(x) U(x, x_0), \\
U(x, x_0) &= I - i\int_{C(x,x_0)} dy_\mu P^\mu_I(y) U(y, x_0), \\
U(x, x_0) &= \mathcal{P} e^{-i\int_C dy_\mu P^\mu_I(y)},
\end{aligned}$$

which is the starting point for covariant perturbation theory (C is a contour in space-time). Note that the four-momentum is not conserved for intermediate states; nor is the four-velocity conserved for intermediate states. Contrast with next section on Bakamjian-Thomas four-momentum operators.

3 Point Form Relativistic Quantum Mechanics

-Start with model Hilbert space of tensor products of single particle spaces, or direct sums thereof. States are written as $|p_i\sigma_i >:= a^\dagger(p_1\sigma_1)...a^\dagger(p_n\sigma_n)|0>$, but under Lorentz transformations, Wigner rotations all different, so cannot couple together.

-Introduce velocity states (Karmanov, NPA644(1998)165; WHK PRC58(1998)3617):

$$\begin{aligned}
|v; k_i, \mu_i >: &= U_{B(v)}|k_i\mu_i >, \sum \vec{k}_i = 0, \\
&= |p_i\sigma_i > \pi D^{j_i}_{\sigma_i\mu_i}(R_w(v, k_i)); \\
U_\Lambda|v; k_i\mu_i > &= |\Lambda v; R_w k_i, \mu_i^{'} > \pi D^{j_i}_{\mu_i^{'}\mu_i}(R_w), \\
M_0|v; k_i\mu_i > &= \sum \sqrt{m_i^2 + \vec{k}_i^2}|v; k_i\mu_i >, \\
P_0^\mu|v; k_i\mu_i > &= v^\mu M_0|v; k_i\mu_i > \\
p_i &= B(v)k_i, R_w = B^{-1}(\Lambda v)\Lambda B(v).
\end{aligned}$$

-Observation: Under a Lorentz transformation $p_i \rightarrow p_i^{'} = \Lambda p_i$ while $k_i \rightarrow k_i^{'} = R_w k_i$; means internal momenta k_i are Wigner rotated under Lorentz transformation. Therefore rotationally invariant kernels of mass operators are Lorentz invariant. A "semi-relativistic" free Hamiltonian, $H_0 = \Sigma\sqrt{m_i^2 + \vec{k}_i^2}$ is Lorentz invariant.

-Total four-momentum operator and Pauli-Lubanski spin operator are

$$\begin{aligned} P^{\mu} &= V^{\mu}(M_0 + M_I) \\ W^{\mu} &= \epsilon^{\mu}_{\nu\alpha\beta} V^{\nu} J^{\alpha\beta}; \\ (M_0 + M_I)|\Psi > &= m|\Psi >, \end{aligned}$$

where interacting mass operator is rotationally invariant and commutes with four-velocity operator. Spin operator depends on kinematic variables only. Four-momentum operator satisfies point form equations.

-Example: Three quark mass operator, called Goldstone boson exchange constituent quark model, with "semirelativistic" kinetic energy, confining potential, and hyperfine interaction, point spectrum with no decays, gives reasonable fit to baryon spectrum (Glozman et al PRD58(1998)094030).

-Example: Take nonrelativistic potential, depends only on internal momenta, use it as interacting mass operator, (Coester, (1973).

-To calculate form factors, also need current operators: If $J^{\mu}(0)$ is the current operator at the space-time point 0 and satisfies

$$\begin{aligned} U_{\Lambda} J^{\mu}(0) U_{\Lambda}^{-1} &= (\Lambda^{\mu}_{\nu})^{-1} J^{\nu}(0) \\ [P^{\mu}, J_{\nu}(0)] &= 0, \end{aligned}$$

then $J^{\mu}(x) := e^{iP\cdot x} J^{\mu}(0) e^{-iP\cdot x}$ at the space-time point x is Poincare covariant and conserved.

-Application: Nucleon electroweak form factors (Graz group, Wagenbrunn, et al, Phy Lett B511(2001)33; Boffi, et al, EPJ A14(2002)17) get excellent agreement with experiment up to several Gev. But note that to correctly normalize charge need Jacobian factors that are like many-body currents (see Melde, et al, PRD76(2007)074020).

-Application: Deuteron form factors: use experimental nucleon form factors, no exchange currents as input; the A structure function falls off too fast, dips in B structure function not in good agreement with experiment, T structure function in reasonable agreement with experiment (Allen, et al PRC63(2001)034002).

- Can also obtain interacting mass operators from velocity state matrix elements of vertex operator(or interaction Lagrangian):

$$M_I \; = \; < v; k_i^{'} \mu_i^{'} | \mathcal{L}(0) | v; k_j \mu_j > f((m^{'} - m)^2)$$

commutes with the four-velocity and is Lorentz invariant. $f()$ depends only on momentum transfer squared and is supposed to compensate for off-diagonal velocity contributions, as well as regulate integrals. M_I produces transitions between direct sum Hilbert spaces.

-Application: Mass operator for pi-N and N-N scattering using chiral perturbation theory. In principle have direct sum spaces with many pions, power counting orders different numbers of pions, use many vertices in mass operator (Girlanda, et al, PRC76(2007)044002).

-Application: Vector meson spectrum with decays, mass operator acts on direct sum Hilbert space, consisting of quark- antiquark and quark-antiquark-pseudoscalar meson with quark-quark-pseudoscalar meson vertex. Quarks and antiquarks bound with harmonic oscillator potential. Eliminating the three particle channel leads to a nonlinear eigenvalue problem, which predicts level shifts and decay widths (Krassnigg et al, PRC67(2003)064003):

$$\begin{aligned}
\begin{bmatrix} M_{q\bar{q}} & K^\dagger \\ K & M_{q\bar{q}\pi} \end{bmatrix} \begin{bmatrix} \Psi_{q\bar{q}} \\ \Psi_{q\bar{q}\pi} \end{bmatrix} &= m \begin{bmatrix} \Psi_{q\bar{q}} \\ \Psi_{q\bar{q}\pi} \end{bmatrix} \\
K\Psi_{q\bar{q}} + M_{q\bar{q}\pi}\Psi_{q\bar{q}\pi} &= m\Psi_{q\bar{q}\pi} \\
\Psi_{q\bar{q}\pi} &= \frac{1}{m - M_{q\bar{q}\pi}} K\Psi_{q\bar{q}} \\
(M_{q\bar{q}} + K^\dagger \frac{1}{m - M_{q\bar{q}\pi}} K)\Psi_{q\bar{q}} &= m\Psi_{q\bar{q}}
\end{aligned}$$

-Application: Alternative way of calculating form factors: consider electron-hadron scattering and include electron in system. Mass operator at hadronic level constructed from electron-photon and hadron-photon vertex; mass operator for electron-quark-antiquark and electron-quark-antiquark-photon (quarks bound with harmonic oscillator potential) from electron-photon vertex, and from quark-photon vertex. Extract form factor by moving electron far from quark-antiquark bound state; raises issue of cluster properties in relativistic quantum mechanics (Biernat et al, PRC79(2009)

4 Point Form S Matrix Cluster Properties

-If have n particle interacting system, break into clusters A and B and move clusters far apart from one another.

-Strong cluster property: The Poincare generators of the total system become additive in the Poincare generators of the A and B systems when A and B are widely separated. Then need packing operators to carry this out (Sokolev, ThMP36(1979)682;
Coester, Polyzou, PRD26(1982)1348).

-Weaker cluster property: The scattering operator for the full system becomes a product of scattering operators for clusters A and B, when A and B are widely separated. (Wichmann, Chrichton PR132(1963)2788).

-Problem in point form: Usually separation operators are (kinematic) spatial translation operators; however, in point form all translation operators are dynamic.

- Since Lorentz transformations are kinematic, use boosts as separation operators. For example, to separate two spinless particles, write

$$\begin{aligned}
U_w^{sep}\phi(p_1,p_2) &= \phi(B^{-1}(w)p_1,p_2); \\
\Psi(x) &= \int \frac{d^3p}{E} e^{-ip\cdot x}\phi(p), \\
U_w^{sep}\Psi(x_1,x_2) &= \Psi(B(w)x_1,x_2), \\
\vec{x} &\rightarrow \vec{x}+\vec{w}t+\vec{w}\frac{\vec{w}\cdot\vec{x}}{w_0+1}
\end{aligned}$$

-Cluster property for two-body kernels:

$$\begin{aligned}
|p_1p_2> &\rightarrow |v,k>, \\
M_0 &= \sqrt{m_1^2+\vec{k}^2}+\sqrt{m_2^2+\vec{k}^2}, \\
<v^{'}k^{'}|M_I|v,k> &= v_0\delta^3(v^{'}-v)K(k^{'},k),
\end{aligned}$$

where $K(k^{'},k)$ is rotationally invariant. Then cluster property is

$$s-lim_{|\vec{w}|\rightarrow\infty}U_w^{sep}M_I(U_w^{sep})^{-1} = 0$$

-Next embed 2 body system in n body system; problem: velocity states with internal momenta k_i do not cluster properly when A and B are widely separated. Instead go to stepwise coupled velocity states:

$$\begin{aligned}
|p_1...p_n> &\rightarrow |v_A, k^A; v_B, k_i^B> \\
&\rightarrow |v, q; k^A, k_i^B>; \\
M_0 &= \sqrt{M_0^2(A)+\vec{q}^2} + \sqrt{M_0^2(B)+\vec{q}^2}; \\
M &= \sqrt{(M_0(A)+M_I(A))^2+\vec{q}^2} + \sqrt{M_0^2(B)+\vec{q}^2} \\
&= M_0 + \tilde{M}_I(A); \\
\tilde{M}_I(A) : &= \sqrt{(M_0(A)+M_I(A))^2+\vec{q}^2} - \sqrt{M_0^2(A)+\vec{q}^2}
\end{aligned}$$

where
$p_1 + p_2 = p_A = B(v)(\sqrt{M_0^2(A)+\vec{q}^2}, \vec{q};$
$p_B = B(v)(\sqrt{M_0^2(B)+\vec{q}^2}, -\vec{q});$
note that when the clusters are separated ($(|\vec{w}| \rightarrow \infty$ the invariant mass $s = (B(w)p_A + p_B)^2$ gets large as does $|\vec{q}|$.

- show that the two body system embedded in the n body system has the same scattering operator as the two body system by itself:

$$\begin{aligned} e^{i(M_0+\tilde{M}_I)\tau}e^{-iM_0\tau} &= e^{i\sqrt{(M_0(A)+M_I(A))^2+\vec{q}^2}\tau}e^{-i\sqrt{M_0^2(A)+\vec{q}^2}\tau} \\ &= e^{i(q+\frac{(M_0(A)+M_I(A))^2}{2q})\tau}e^{-i(q+\frac{M_0^2(A)}{2q})\tau}, \end{aligned}$$

when $q := |\vec{q}|$ is large.

-Consider a four-body system and use the same construction for all the two-body forces in the 4-body system. Then get a mass operator of the form

$$M = M_0 + \tilde{M}_I(1-2) + \tilde{M}_I(1-3) + ...\tilde{M}_I(3-4).$$

-Break an n-body system into clusters A and B. The four-momentum of a particle in cluster A will be written $p_i, i \in A$, similarly for particles in cluster B. Then the total momentum of particles in cluster A is

$$\begin{aligned} p_A &= \sum_{i \in A} p_i \\ &= B(v) q_A, q_A = (\sqrt{M_0^2(A) + \vec{v}^2}, \vec{q}); \\ p_i &= B(v_A) k_i^A, i \in A, v_A = \frac{p_A}{M_0(A)} \\ &= B(v) B(\frac{q_A}{\sqrt{s_A}}) R^{-1}(B(v), q_A) k_i^A, \end{aligned}$$

where $\vec{q}$ is the momentum between the A and B clusters.

The invariant "mass" squared of the 4 (or n) particle system is

$$s = (p_A + p_B)^2 = (\sqrt{M_0^2(A) + q^2} + \sqrt{M_0^2(B) + q^2})^2$$

and when q gets large, s gets large. It follows that subenergies $s_{ij} = (p_i + p_j)^2$ will only get large if i is in the A cluster and j is in the B cluster. If i and j are in the same cluster the subenergy remains finite when q gets large.

Then in the limit when q gets large, the 4-body scattering operator becomes

$$
\begin{aligned}
e^{iM\tau}e^{-iM_0\tau} &= e^{i(\sqrt{M_0^2(A)+q^2}+\sqrt{M_0^2(B)+q^2}+\sum \tilde{M}_I(i-j))\tau} \\
&\quad \times e^{-i(\sqrt{M_0^2(A)+q^2}+\sqrt{M_0^2(B)+q^2})\tau} \\
&= e^{i(q+\frac{M_0^2(A)}{2q}+q+\frac{M_0^2(B)}{2q}+\tilde{M}(A)+\tilde{M}(B))\tau} \\
&\quad \times e^{-i(q+\frac{M_0^2(A)}{2q}+q+\frac{M_0^2(B)}{2q})\tau} \\
&= e^{i(\frac{(M_0(A)+M_I(A))^2}{2q}+\frac{(M_0(B)+M_I(B))^2}{2q})\tau} e^{-i(\frac{M_0^2(A)}{2q}+\frac{M_0^2(B)}{2q})\tau} \\
&= e^{i\frac{(M_0(A)+M_I(A))^2}{2q}\tau} e^{-i\frac{M_0^2(A)}{2q}\tau} \\
&\quad \times e^{i\frac{(M_0(B)+M_I(B))^2}{2q})\tau} e^{-i\frac{M_0^2(B)}{2q})\tau}
\end{aligned}
$$

Conclusion: In stepwise coupled velocity state variables, the 4-body scattering operator becomes a product of A and B cluster scattering operators when q is large.

-Application: Electron scattering on hadrons, with the electron included in the mass operator. In the limit as q (or s) gets large the scattering amplitude becomes the hadronic form factor(Biernat et al).

5 Conclusion-Open Questions

-Gluon Vacuum: Is it possible to construct a third order polynomial S in gluon fields such that

$$|\Omega > \ = \ e^{S}|0 >$$

transforms away the gluon self energy?

-Strong Coupling Model: Neglecting the nucleon kinetic energy can one construct an automorphism (on the algebra of operators) on the pion fields such that the four-momentum operator is diagonalized with a Bogoliubov transformation? Then include the nucleon kinetic energy perturbatively.

- Bakamjian-Thomas like four-momentum operator: Again neglecting nucleon kinetic energy construct a four-momentum operator satisfying the point form equations, where the coupling of nucleons to pions is given phenomenologically and diagonalize with a Bogoliubov transformation.

-Decay Widths: Augment mass operators that fit baryon spectrum (point spectrum) by including meson decay channels in a coupled channel approach with the off diagonal mass operator given by velocity state matrix elements of vertex operators.

-Form Factors: Include electron in mass operator and extract form factor in suitable cluster limit; include vertices that give rise to exchange currents.

-Velocity State "Racah Coefficients": Calculate the coefficients that transform between different stepwise coupled velocity states.

-Cluster Properties: Use Lorentz boosts as separation operators to demonstrate that the scattering operator becomes a product when two clusters are widely separated. Demonstrate for n-particle systems including spin, particle production, identical particles and bound states.